3 Exercises

Problem 1. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & 3x_1 + x_2 + x_3 = 5 \\ & x_1 + x_2 + x_3 = 1 \end{aligned}$$

$$L(\lambda_1, \lambda_2, x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - \lambda_1[3x_1 + x_2 + x_3 - 5] - \lambda_2[x_1 + x_2 + x_3 - 1]$$

$$\nabla L(\lambda_1, \lambda_2, x_1, x_2, x_3) = \begin{bmatrix} -(3x_1 + x_2 + x_3 - 5) \\ -(x_1 + x_2 + x_3 - 1) \\ 2x_1 - 3\lambda_1 - \lambda_2 \\ 2x_2 - \lambda_1 - \lambda_2 \\ 2x_3 - \lambda_1 - \lambda_2 \end{bmatrix} \qquad H_L(\lambda_1, \lambda_2, x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & -3 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ -3 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$\nabla L = 0$:

$$3x_1 + x_2 + x_3 = 5 \quad (1)$$
$$x_1 + x_2 + x_3 = 1 \quad (2)$$
$$2x_1 = 3\lambda_1 + \lambda_2 \quad (3)$$
$$2x_2 = \lambda_1 + \lambda_2 \quad (4)$$
$$2x_3 = \lambda_1 + \lambda_2 \quad (5)$$

$(3) \Rightarrow x_1 = \frac{3\lambda_1 + \lambda_2}{2}$

$(4) \Rightarrow x_2 = \frac{\lambda_1 + \lambda_2}{2}$

$(5) \Rightarrow x_3 = \frac{\lambda_1 + \lambda_2}{2}$

with $(1), (2)$ $\Rightarrow$ $11\lambda_1 + 5\lambda_2 = 10$, $5\lambda_1 + 3\lambda_2 = 2$

Cramer's rule

$\Rightarrow \lambda_1 = \frac{20}{8} = \frac{5}{2} \qquad \lambda_2 = -\frac{28}{8} = -\frac{7}{2}$

$(3),(4),(5) \Rightarrow x_1 = 2, \quad x_2 = -\frac{1}{2}, \quad x_3 = -\frac{1}{2}$

$\Rightarrow$ CCPs: $\left(\frac{5}{2}, -\frac{7}{2}, 2, -\frac{1}{2}, -\frac{1}{2}\right)$

2^{nd} deriv. test:

$\left(\frac{5}{2}, -\frac{7}{2}, 2, -\frac{1}{2}, -\frac{1}{2}\right)$:

$$H_L\left(\frac{5}{2}, -\frac{7}{2}, 2, -\frac{1}{2}, -\frac{1}{2}\right) = \begin{bmatrix} 0 & 0 & -3 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ -3 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$k = 2 \Rightarrow 2k + 1 = 5$

So, $d_5 = \left|H_L\left(\frac{5}{2}, -\frac{7}{2}, 2, -\frac{1}{2}, -\frac{1}{2}\right)\right| = 16$

$\Rightarrow (-1)^k d_5 > 0$

$\Rightarrow f$ has a constrained local min. at $\left(2, -\frac{1}{2}, -\frac{1}{2}\right)$